

## ON DIOPHANTINE QUINTUPLE CONJECTURE

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ABSTRACT. In this note, we prove that if  $\{a, b, c, d, e\}$  with  $a < b < c < d < e$  is a Diophantine quintuple, then  $d < 10^{76}$ .

A set of  $m$  distinct positive integers  $\{a_1, \dots, a_m\}$  is called a Diophantine  $m$ -tuple if  $a_i a_j + 1$  is a perfect square. Diophantus studied sets of positive rational numbers with the same property, particularly he found the set of four positive rational numbers  $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ . But the first Diophantine quadruple was found by Fermat. In fact, Fermat proved that the set  $\{1, 3, 8, 120\}$  is a Diophantine quadruple called *Fermat set*. Moreover, Baker and Davenport [1] proved that the set  $\{1, 3, 8, 120\}$  cannot be extended to a Diophantine quintuple.

Several results of the generalization of the result of Baker and Davenport are obtained. In 1997, Dujella [2] proved that the Diophantine triples of the form  $\{k-1, k+1, 4k\}$ , for  $k \geq 2$ , cannot be extended to a Diophantine quintuple. The Baker-Davenport's result corresponds to  $k = 2$ . In 1998, Dujella and Pethő [4] proved that the Diophantine pair  $\{1, 3\}$  cannot be extended to a Diophantine quintuple. In 2008, Fujita [8] obtained a more general result by proving that the Diophantine pairs  $\{k-1, k+1\}$ , for  $k \geq 2$ , cannot be extended to a Diophantine quintuple. A folklore conjecture is

**Conjecture.** *There does not exist a Diophantine quintuple.*

In 2004, Dujella [5] proved that there are only finitely many Diophantine quintuples. Assuming that  $\{a, b, c, d, e\}$  is a Diophantine quintuple with  $a < b < c < d < e$ , authors got the upper bound of element  $d$ :

- i)  $d < 10^{2171}$ , Dujella [5];
- ii)  $d < 10^{830}$ , Fujita [9];
- iii)  $d < 10^{100}$ , Filipin and Fujita [10].
- iv)  $d < 3.5 \cdot 10^{94}$ , Elsholtz, Filipin and Fujita [7].

Moreover, by using upper bound of  $d$ , corresponding upper bound of number of Diophantine quintuples are obtained,  $10^{1930}$ ,  $10^{276}$ ,  $10^{96}$  and  $6.8 \cdot 10^{32}$  respectively.

In this paper, we prove the following results.

**Theorem 1.** *If  $\{a, b, c, d, e\}$  is a Diophantine quintuple with  $a < b < c < d < e$ , then  $d < 10^{76}$ .*

From now on, we will assume that  $\{a, b, c, d, e\}$  is a Diophantine quintuple with  $a < b < c < d < e$ . Let us consider a Diophantine triple  $\{A, B, C\}$ . We define

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the positive integers  $R, S, T$  by

$$AB + 1 = R^2, \quad AC + 1 = S^2, \quad BC + 1 = T^2.$$

In order to extend the Diophantine triple  $\{A, B, C\}$  to a Diophantine quadruple  $\{A, B, C, D\}$ , we have to solve the system

$$AD + 1 = x^2, \quad BD + 1 = y^2, \quad CD + 1 = z^2,$$

in integers  $x, y, z$ . Eliminating  $D$ , we obtain the following system of Pellian equations.

$$(1) \quad Az^2 - Cx^2 = A - C,$$

$$(2) \quad Bz^2 - Cy^2 = B - C.$$

All solutions of (1) and (2) are respectively given by  $z = v_m$  and  $z = w_n$  for some integer  $m, n \geq 0$ , where

$$v_0 = z_0, \quad v_1 = Sz_0 + Cx_0, \quad v_{m+2} = 2Sv_{m+1} - v_m,$$

$$w_0 = z_1, \quad w_1 = Tz_1 + Cy_1, \quad w_{n+2} = 2Tw_{n+1} - w_n,$$

with some integers  $z_0, z_1, x_0, y_1$ .

By Lemma 3 of [5], we have the following relations between  $m$  and  $n$ .

**Lemma 1.** *If  $v_{2m} = w_{2n}$ , then  $n \leq m \leq 2n$ .*

We will give a new lower bound of  $m$ . this paper.

**Lemma 2.** *If  $B \geq 8$  and  $v_{2m} = w_{2n}$  has solutions  $m \geq 3, n \geq 2$ , then  $m > 0.48B^{-1/2}C^{1/2}$ .*

*Proof.* By Lemma 4 in [3] and  $z_0 = z_1 = \lambda \in \{1, -1\}$ , we have

$$Am^2 + \lambda Sm \equiv Bn^2 + \lambda Tn \pmod{4C}.$$

Suppose that  $m \leq 0.48B^{-1/2}C^{1/2}$ . From the relation  $n \leq m$ , we get

$$\max\{Am^2, Bn^2\} \leq Bm^2 \leq 0.25B \cdot B^{-1}C < 0.25C$$

and

$$\max\{Sm, Tn\} \leq Tm < 0.48(BC+1)^{1/2}B^{-1/2}C^{1/2} < 0.5(BC)^{1/2}B^{-1/2}C^{1/2} = 0.5C.$$

We obtain that

$$Am^2 - Bn^2 = \lambda(Tn - Sm).$$

This implies

$$\begin{aligned} \lambda(Tn + Sm)(Am^2 - Bn^2) &= T^2n^2 - S^2m^2 \\ &= (BC + 1)n^2 - (AC + 1)m^2 = C(Bn^2 - Am^2) + n^2 - m^2. \end{aligned}$$

It follows that

$$m^2 - n^2 = (C + \lambda(Tn + Sm))(Bn^2 - Am^2).$$

If  $Bn^2 - Am^2 = 0$ , then  $m = n$ , it is impossible. Hence,

$$m^2 - n^2 = |m^2 - n^2| \geq |C + \lambda(Tn + Sm)|.$$

The case  $\lambda = 1$  provides  $m^2 > C$ , it is a contradiction to  $m < 0.48B^{-1/2}C^{1/2}$ . From  $Tn + Sm < 2Tn < C$ , we need to consider

$$m^2 - n^2 = |m^2 - n^2| \geq |C - (Tn + Sm)| = C - (Tn + Sm).$$

Therefore, we get the inequality

$$\begin{aligned} C &\leq Tn + Sm + m^2 - n^2 \leq 2Tm + 0.75m^2 \\ &< 0.96(BC + 1)^{1/2}B^{-1/2}C^{1/2} + 0.173B^{-1}C < C \end{aligned}$$

when  $B \geq 8$ . We have a contradiction. This completes the proof.  $\square$

**Proof of Theorem 1.** Assume that  $\{a, b, c, d, e\}$  is a Diophantine quintuple with  $a < b < c < d < e$ . In [4], Dujella and Pethö have shown that the pair  $\{1, 3\}$  cannot extend to a Diophantine quintuple. This help us to assume that  $b \geq 8$ .

We choose that

$$A = a, B = b, C = d, D = e$$

in the Diophantine quintuple  $\{a, b, c, d, e\}$ . This implies the system of Pellian equations (1) and (2) has positive integer solution  $(x, y, z)$  with  $|z| > 1$ . Equivalently, there are positive integers  $j$  and  $k$  satisfying  $v_j = w_k$ . By Lemma 5 and Lemma 6 of [9], we have  $j \equiv k \equiv 0 \pmod{2}$ ,  $k \geq 4$ ,  $z_0 = z_1 = \pm 1$ . We set  $j = 2m$  and  $k = 2n$ . Using Lemma 2, we have  $m \geq 0.48B^{-1/2}C^{1/2}$ .

It is known that  $d \geq d^+ > 4abc > 4b^2$ , where  $d^+ = a + b + c + 2abc + 2rst$ . It results  $B = b < d^{1/2}/2 = C^{1/2}/2$ . Hence, we have

$$(3) \quad m \geq 0.678C^{1/4}.$$

On the other hand, by used Theorem 2.1 in [11] of Matveev, we have the relative upper bound (cf. Proposition 14 of [9])

$$(4) \quad \frac{m}{\log(351m)} < 2.786 \cdot 10^{12} \cdot \log^2 C.$$

Combining (3) and (4), we obtain

$$C^{1/4} < 4.11 \cdot 10^{12} \cdot \log^2 C \cdot \log(238C^{1/4}).$$

Therefore, we have  $d = C < 10^{76}$ . This complete Theorem 1.  $\square$

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